BMO functions and Carleson measures with values in uniformly convex spaces

Caiheng Ouyang and Quanhua Xu

Abstract

This paper studies the relationship between vector-valued BMO functions and the Carleson measures defined by their gradients. Let dA and dm denote Lebesgue measures on the unit disc D and the unit circle $\mathbb{T}$, respectively. For $1 < q < \infty$ and a Banach space B we prove that there exists a positive constant c such that

$$\sup_{z_0\in D}\int_D (1-|z|)^{q-1}\|\nabla f(z)\|^q P_{z_0}(z)\,dA(z) \le c^q \sup_{z_0\in D}\int_{\mathbb{T}} \|f(z)-f(z_0)\|^q P_{z_0}(z)\,dm(z)$$

holds for all trigonometric polynomials f with coefficients in B iff B admits an equivalent norm which is q-uniformly convex, where

$$P_{z_0}(z) = \frac{1-|z_0|^2}{|1-\bar z_0 z|^2}\,.$$

The validity of the converse inequality is equivalent to the existence of an equivalent q-uniformly smooth norm.

0 Introduction

Let $\mathbb{T}$ be the unit circle of the complex plane equipped with normalized Haar measure dm. Recall that an integrable function f on $\mathbb{T}$ is of bounded mean oscillation (BMO) if

$$\|f\|_* = \sup_I \frac{1}{|I|}\int_I |f-f_I|dm < \infty,$$

where the supremum runs over all arcs of $\mathbb{T}$ and $f_I = |I|^{-1}\int_I f dm$ is the mean of f over I. Let $BMO(\mathbb{T})$ denote the space of BMO functions on $\mathbb{T}$. The means over arcs in this definition can be replaced by the averages of f against the Poisson kernel P_{z_0} for the unit disc D:

$$P_{z_0}(z) = \frac{1-|z_0|^2}{|1-\bar z_0 z|^2}\,,\quad z_0\in D,\, z\in\mathbb{T}.$$

Then

$$\|f\|_*^2 \approx \sup_{z_0\in D}\int_{\mathbb{T}} |f(z)-f(z_0)|^2 P_{z_0}(z)dm(z)$$

C.O.: Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences P.O. Box 71010 Wuhan 430071, China
ouyang@wipm.ac.cn
Q.X.: Laboratoire de Mathématiques, Université de France-Comté, 25030 Besançon Cedex, France
qxu@univ-fcomte.fr
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with universal equivalence constants. Here as well as in the sequel, we denote also by f its Poisson integral in D:

$$f(z_0) = \int_{\mathbb{T}} f(z) P_{z_0}(z) dm(z), \quad z_0 \in D.$$

On the other hand, it is classical that BMO functions can be characterized by Carleson measures. A positive measure μ on D is called a Carleson measure if

$$\|\mu\|_C = \sup_{z_0 \in D} \int_D \frac{1-|z_0|^2}{|1-\bar{z}_0 z|^2}\, d\mu(z) < \infty.$$

Let $f \in L^1(\mathbb{T})$. Then $f \in MBO(\mathbb{T})$ iff $|\nabla f(z)|^2\,(1-|z|^2)\,dA(z)$ is a Carleson measure, where $dA(z)$ denotes Lebesgue measure on D. In this case, we have

$$\|f\|_*^2 \approx \sup_{z_0 \in D} \int_D |\nabla f(z)|^2\, \frac{(1-|z|^2)(1-|z_0|^2)}{|1-\bar{z}_0 z|^2}\, dA(z). \tag{0.1}$$

We refer to [6] for all these results.

This paper concerns the vector-valued version of (0.1). More precisely, we are interested in characterizing Banach spaces B for which one of the two inequalities in (0.1) holds for B-valued functions f. Given a Banach space B let $L^p(\mathbb{T}; B)$ denote the usual L^p-space of Bochner p-integrable functions on $\mathbb{T}$ with values in B. The space $BMO(\mathbb{T}; B)$ of B-valued functions on $\mathbb{T}$ is defined in the same way as in the scalar case just by replacing the absolute value of $\mathbb{C}$ by the norm of B. Then the vector-valued analogue of (0.1) is the following:

$$c_1^{-1}\,\|f\|_*^2 \le \sup_{z_0 \in D} \int_D \|\nabla f(z)\|^2\, \frac{(1-|z|^2)(1-|z_0|^2)}{|1-\bar{z}_0 z|^2}\, dA(z) \le c_2\,\|f\|_*^2 \tag{0.2}$$

for all $f \in BMO(\mathbb{T}; B)$, where c_1, c_2 are two positive constants (depending on B), and where

$$\|\nabla f(z)\| = \Big\|\frac{\partial f}{\partial x}(z)\Big\| + \Big\|\frac{\partial f}{\partial y}(z)\Big\|, \quad z = x + iy.$$

It is part of the folklore that (0.2) holds iff B is isomorphic to a Hilbert space (see [2]). We include a proof of this result at the end of the paper for the convenience of the reader.

However, if one considers the validity of only one of the two inequalities in (0.2), the matter becomes much subtler and the corresponding class of Banach spaces is much larger. The following theorem solves this problem.

Theorem 0.1 *Let B be a Banach space.*

(i) *There exists a positive constant c such that*

$$\sup_{z_0 \in D} \int_D \|\nabla f(z)\|^2\, \frac{(1-|z|^2)(1-|z_0|^2)}{|1-\bar{z}_0 z|^2}\, dA(z) \le c\,\|f\|_*^2$$

holds for all trigonometric polynomials f with coefficients in B iff B admits an equivalent norm which is 2-uniformly convex.

(ii) *There exists a positive constant c such that*

$$\sup_{z_0 \in D} \int_D \|\nabla f(z)\|^2\, \frac{(1-|z|^2)(1-|z_0|^2)}{|1-\bar{z}_0 z|^2}\, dA(z) \ge c^{-1}\,\|f\|_*^2$$

for all trigonometric polynomials f with coefficients in B iff B admits an equivalent norm which is 2-uniformly smooth.

We refer to the next section for the definition of uniform convexity (smoothness). This theorem is intimately related to the main result of [15], where the vector-valued Littlewood-Paley theory is studied. Given $f \in L^1(\mathbb{T}; B)$ define the Littlewood-Paley g-function

$$\big(G(f)(z)\big)^2 = \int_0^1 (1-r)\,\|\nabla f(rz)\|^2\,dr\,, \quad z \in \mathbb{T}.$$

The following fact is again well-known: the equivalence

$$\big\|G(f)\big\|_{L^2(\mathbb{T})} \approx \|f - f(0)\|_{L^2(\mathbb{T};B)}$$

holds uniformly for all B-valued trigonometric polynomials f iff B is isomorphic to a Hilbert space. However, the two one-sided inequalities are related to uniform convexity (smoothness). More precisely, we have the following result from [15].

Theorem 0.2 *Let B be a Banach space.*

(i) *B has an equivalent 2-uniformly convex norm iff for some $p \in (1,\infty)$ (or equivalently, for every $p \in (1,\infty)$) there exists a positive constant c such that*

$$\big\|G(f)\big\|_{L^p(\mathbb{T})} \le c\,\|f\|_{L^p(\mathbb{T};B)} \tag{0.3}$$

holds for all B-valued trigonometric polynomials f.

(ii) *B has an equivalent 2-uniformly smooth norm iff for some $p \in (1,\infty)$ (or equivalently, for every $p \in (1,\infty)$) there exists a positive constant c such that*

$$\|f - f(0)\|_{L^p(\mathbb{T};B)} \le c\,\big\|G(f)\big\|_{L^p(\mathbb{T})} \tag{0.4}$$

holds for all B-valued trigonometric polynomials f.

According to [15], the spaces satisfying (0.3) (resp. (0.4)) are said to be of Lusin cotype 2 (resp. Lusin type 2). The name Lusin refers to the fact that the Littlewood-Paley g-function can be replaced by the Lusin area function. At this stage, let us also recall that by Pisier's renorming theorem [10], B has an equivalent 2-uniformly convex (resp. smooth) norm iff B is of martingale cotype (resp. type) 2

The value $p = \infty$ is, of course, not allowed in Theorem 0.2. At the time of the writing of [15], the second named author guessed that a right substitute of Theorem 0.2 for $p = \infty$ should be Theorem 0.1 but could not confirm this. Our proof of Theorem 0.1 heavily relies on Theorem 0.2 and Calderón-Zygmund singular integral theory. In fact, we will work in the more general setting of an Euclidean space $\mathbb{R}^n$ instead of $\mathbb{T}$. On the other hand, the power 2 in $\|\nabla f\|^2$ plays no longer any special role in the vector-valued setting. We will consider the analogue of Theorem 0.1 for $\|\nabla f\|^q$ with $1 < q < \infty$. The corresponding result is stated separately in Theorems 3.1 and 4.1 below, which correspond to the end point $p = \infty$ of the results of [9] and [15].

1 Preliminaries

Our references for harmonic analysis are [5], [6] and [14]. All results quoted in the sequel without explicit reference can be found there. However, one needs sometimes to adapt arguments in the scalar case to the vector-valued setting.

Let (Ω, μ) be a measure space and B a Banach space. For $1 \le p \le \infty$ we denote by $L^p(\Omega, \mu; B)$ the usual L^p-space of Bochner (or strongly) measurable functions on Ω with values in B. The norm of $L^p(\Omega, \mu; B)$ is denoted by $\|\ \|_p$. The n-dimensional Euclidean space $\mathbb{R}^n$ is equipped with

Lebesgue measure. $L^1_{\rm loc}(\mathbb{R}^n; B)$ denotes the space of locally integrable functions on $\mathbb{R}^n$ with values in B. Recall the Poisson kernel on $\mathbb{R}^n$:

$$P_t(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}\,\frac{t}{(t^2+|x|^2)^{(n+1)/2}}\,, \quad x\in\mathbb{R}^n\,,\ t>0.$$

Let $f\in L^1_{\rm loc}(\mathbb{R}^n; B)$ such that

$$\int_{\mathbb{R}^n}\|f(x)\|\,\frac{1}{1+|x|^{n+1}}\,dx<\infty.$$

The Poison integral of f is then defined by

$$P_t * f(x) = \int_{\mathbb{R}^n} P_t(x-y)f(y)\,dy.$$

The function $P_t * f(x)$ is harmonic in the upper half space $\mathbb{R}^{n+1}_+$. Let us make a convention used throughout this paper: For a function $f\in L^1_{\rm loc}(\mathbb{R}^n; B)$ we also denote by f its Poisson integral (whenever the latter exists); thus $f(x,t) = P_t * f(x)$.

The space $BMO(\mathbb{R}^n; B)$ is defined as the space of all functions $f\in L^1_{\rm loc}(\mathbb{R}^n; B)$ such that

$$\|f\|_* = \sup_Q \frac{1}{|Q|}\int_Q\|f(x)-f_Q\|\,dx<\infty,$$

where the supremum runs over all cubes $Q\subset\mathbb{R}^n$ (with sides parallel to the axes), and where f_Q denotes the mean of f over Q. Equipped with $\|\ \|_*$, $BMO(\mathbb{R}^n; B)$ is a Banach space modulo constants. $BMO(\mathbb{R}^n;\mathbb{C})$ is simply denoted by $BMO(\mathbb{R}^n)$.

We will also need the Hardy space H^1. There exist several different (equivalent) ways to define this. It is more convenient for us to use atomic decomposition. A B-valued atom is a function $a\in L^\infty(\mathbb{R}^n; B)$ such that

$$\mathrm{supp}(a)\subset Q,\quad \int_{\mathbb{R}^n} a\,dx = 0,\quad \|a\|_\infty\le\frac{1}{|Q|}$$

for some cube $Q\subset\mathbb{R}^n$. We then define $H^1_a(\mathbb{R}^n; B)$ to be the space of all functions f which can be written as

$$f=\sum_{k\ge1}\lambda_k\,a_k$$

with a_k atoms and λ_k scalars such that $\sum_k|\lambda_k|<\infty$. The norm of $H^1_a(\mathbb{R}^n; B)$ is defined by

$$\big\|f\big\|_{H^1_a(\mathbb{R}^n;B)} = \inf\big\{\sum_{k\ge1}|\lambda_k|\ :\ f=\sum_{k\ge1}\lambda_k\,a_k\big\}.$$

This is a Banach space. It is well-known that $H^1_a(\mathbb{R}^n; B)$ coincides with the space of all $f\in L^1(\mathbb{R}^n; B)$ such that

$$\sup_{t>0}\|f(\,\cdot,t)\|\in L^1(\mathbb{R}^n).$$

Fefferman's duality theorem between H^1 and BMO remains valid in this setting (with a slight condition on B). More precisely, $BMO(\mathbb{R}^n; B^*)$ is isomorphically identified as a subspace of the dual $H^1_a(\mathbb{R}^n; B)^*$; moreover, it is norming in the following sense: for any $f\in H^1_a(\mathbb{R}^n; B)$

$$\big\|f\big\|_{H^1_a(\mathbb{R}^n;B)}\approx\sup\big\{|\langle f,\ g\rangle|\ :\ g\in BMO(\mathbb{R}^n; B^*),\ \|g\|_*\le1\big\}$$

with universal equivalence constants. Note that this duality result follows immediately from the atomic definition of H^1_a. If B^* has the Radon-Nikodym property (in particular, if B is reflexive), then

$$H^1_a(\mathbb{R}^n; B)^* = BMO(\mathbb{R}^n; B^*).$$

We refer to [1] and [3] for more details.

BMO functions can be characterized by Carleson measures. Let

$$\Gamma = \{(z,t) \in \mathbb{R}^{n+1}_+ \ : \ |z| < t\},$$

the standard cone of $\mathbb{R}^{n+1}_+$. $\Gamma(x)$ denotes the translation of Γ by $(x,0)$ for $x \in \mathbb{R}^n$: $\Gamma(x) = \Gamma + (x,0)$. Let Q be a cube. The tent over Q is defined by

$$\widehat{Q} = \mathbb{R}^{n+1}_+ \setminus \bigcup_{x \in Q^c} \Gamma(x)\,.$$

A positive measure μ on $\mathbb{R}^{n+1}_+$ is called a Carleson measure if

$$\|\mu\|_C = \sup_{Q \text{ cube}} \frac{\mu(\widehat{Q})}{|Q|} < \infty.$$

Then $f \in BMO(\mathbb{R}^n)$ iff $\mu(f) = (t|\nabla f(x,t)|)^2 dxdt/t$ is a Carleson measure. Moreover,

$$\|f\|_*^2 \approx \|\mu(f)\|_C.$$

This is the analogue of (0.1) for $\mathbb{R}^n$. Our main concern is the validity of each of the two one-sided inequalities of the equivalence above in the vector-valued setting. The previous result is, of course, part of the Littlewood-Paley theory. In this regard let us recall its L^p-analogue. Let $f \in L^p(\mathbb{R}^n)$. Define the Lusin integral function of f:

$$\big(S(f)(x)\big)^2 = \int_\Gamma (t|\nabla f(x+z,t)|)^2 \,\frac{dzdt}{t^{n+1}}\,, \quad x \in \mathbb{R}^n\,.$$

Then

$$\|f\|_p \approx \|S(f)\|_p\,, \quad \forall\ f \in L^p(\mathbb{R}^n),\ 1 < p < \infty.$$

The vector-valued Littlewood-Paley theory is studied in [9] and [15]. Let $1 < q < \infty$ and $f \in L^p(\mathbb{R}^n; B)$. Define

$$\big(S_q(f)(x)\big)^q = \int_\Gamma (t\|\nabla f(x+z,t)\|)^q \,\frac{dzdt}{t^{n+1}}\,, \quad x \in \mathbb{R}^n\,,$$

where

$$\|\nabla f(x,t)\| = \Big\|\frac{\partial}{\partial t} f(x,t)\Big\| + \sum_{i=1}^n \Big\|\frac{\partial}{\partial x_i} f(x,t)\Big\|.$$

According to [15] and [9], B is said to be of Lusin cotype q if for some $p \in (1,\infty)$ (or equivalently, for every $p \in (1,\infty)$) there exists a positive constant c such that

$$\big\|S_q(f)\big\|_p \le c\,\big\|f\|_p$$

for all compactly supported B-valued continuous functions f on $\mathbb{R}^n$. Similarly, we define Lusin type q by reversing the inequality above. Note that if B is of Lusin cotype (resp. type) q, then necessarily $q \ge 2$ (resp. $q \le 2$). By [15] and [9], Lusin cotype (resp. type) q is equivalent to martingale cotype (resp. type) q. We will not need the latter notion and refer the interested reader to [11] and [10]. By Pisier's renorming theorem [10], B is of martingale cotype (resp. type) q iff B has an equivalent norm which is q-uniformly convex (resp. smooth). Let us recall this last notion

for which we refer to [8] for more information. First define the modulus of convexity and modulus of smoothness of B by

$$\begin{aligned}
\delta_B(\varepsilon) &= \inf\big\{1-\big\|\frac{a+b}{2}\big\| \;:\; a,\ b\in B,\ \|a\|=\|b\|=1,\ \|a-b\|=\varepsilon\big\},\quad 0<\varepsilon<2,\\
\rho_B(t) &= \sup\big\{\frac{\|a+tb\|+\|a-tb\|}{2}-1 \;:\; a,\ b\in B,\ \|a\|=\|b\|=1\big\},\quad t>0.
\end{aligned}$$

B is called uniformly convex if $\delta_B(\varepsilon)>0$ for every $\varepsilon>0$, and uniformly smooth if $\lim_{t\to 0}\rho_B(t)/t=0$. On the other hand, if $\delta_B(\varepsilon)\ge c\,\varepsilon^q$ for some positive constants c and q, B is called q-uniformly convex. Similarly, we define q-uniformly smoothness by demanding $\rho_B(t)\le c\,t^q$ for some $c>0$ and $q>1$. It is well-known that for $1<p<\infty$ any (commutative or noncommutative) L^p-space is $\max(2,p)$-uniformly convex and $\min(2,p)$-uniformly smooth.

2 A singular integral operator

Let the cone $\Gamma=\{(z,t)\in\mathbb{R}^n_+ \;:\; |z|<t\}$ be equipped with the measure $dzdt/t^{n+1}$. Let $1<q<\infty$ and B be a Banach space. Set $A=L^q(\Gamma;B)$. For $h\in L^p(\mathbb{R}^n;A)$ we will consider h as a function of either a sole variable $x\in\mathbb{R}^n$ or three variables $(x,z,t)\in\mathbb{R}^n\times\Gamma$. In the first case $h(x)$ is a function of two variables (z,t) for every $x\in\mathbb{R}^n$. Thus $h(x)(z,t)=h(x,z,t)$.

We will consider singular integral operators with kernels taking values in $\mathcal{L}(A)$, the space of bounded linear operators on A. Recall that P_t denotes the Poisson kernel on $\mathbb{R}^n$. Let

$$\varphi_t(x)=t\frac{\partial}{\partial t}P_t(x). \tag{2.1}$$

For $h\in L^p(\mathbb{R}^n;A)$ define

$$\Phi(h)(x,u,s)=\int_\Gamma\int_{\mathbb{R}^n}\varphi_s*\varphi_t(x+u+z-y)h(y,z,t)\,dy\,\frac{dzdt}{t^{n+1}}. \tag{2.2}$$

$\Phi(h)$ is well defined for h in a dense vector subspace of $L^p(\mathbb{R}^n;A)$. Indeed, let $h:\mathbb{R}^n\to A$ be a compactly supported continuous function such that for each $x\in\mathrm{supp}(h)$ the function $h(x):\Gamma\to B$ is continuous and supported by a compact of Γ independent of x. Then it is easy to check that $\Phi(h)$ is well defined and belongs to $L^p(\mathbb{R}^n;A)$ for all p. On the other hand, it is clear that the family of all such functions h is dense in $L^p(\mathbb{R};A)$ for every $p<\infty$. In the sequel, h will be assumed to belong to this family whenever we consider $\Phi(h)$.

The following will be crucial later on. We refer to [15] for a similar lemma on the circle $\mathbb{T}$.

Lemma 2.1 *The map Φ extends to a bounded map on $L^p(\mathbb{R}^n;A)$ for every $1<p<\infty$, and also a bounded map from $H^1_a(\mathbb{R}^n;A)$ to $L^1(\mathbb{R}^n;A)$. Moreover, denoting again by Φ the extended maps, we have*

$$\|\Phi: L^p(\mathbb{R}^n;A)\to L^p(\mathbb{R}^n;A)\|\le c\,,\quad \|\Phi: H^1_a(\mathbb{R}^n;A)\to L^1(\mathbb{R}^n;A)\|\le c\,,$$

where the constant c depends only on p,q and n.

A similar statement holds for each of the n partial derivatives in x_i instead of $\partial/\partial t$ in the definition of φ in (2.1).

Proof. The proof is based on Calderón-Zygmund singular integral theory for vector-valued kernels for which we refer to [5]. We will represent Φ as a singular integral operator. Let

$$k_{s,t}(x)=\varphi_s*\varphi_t(x)=\int_{\mathbb{R}^n}\varphi_s(x-y)\varphi_t(y)dy.$$

Then

$$\Phi(h)(x,u,s)=\int_\Gamma\int_{\mathbb{R}^n}k_{s,t}(x+u+z-y)h(y,z,t)\,dy\,\frac{dzdt}{t^{n+1}}. \tag{2.3}$$

On the other hand, using the definition of φ_t and the semigroup property of P_t, we find

$$k_{s,t}(x) = st\frac{\partial^2}{\partial r^2}P_r(x)\big|_{r=s+t}\,. \tag{2.4}$$

Now consider the operator-valued kernel $K(x) : A \to A$ defined by

$$K(x)(a)(u,s) = \int_\Gamma k_{s,t}(x+u+z)a(z,t)\frac{dzdt}{t^{n+1}}\,,\quad a \in A.$$

Then $\Phi(h)$ can be rewritten as

$$\Phi(h)(x) = K * h(x) = \int_{\mathbb{R}^n} K(x-y)(h(y))dy.$$

Thus Φ is a convolution operator with kernel K. We will show that K is a regular Calderón-Zygmund kernel with values in $\mathcal{L}(A)$. Namely, K satisfies the following norm estimates

$$\|K(x)\| \le \frac{c}{|x|^n} \quad \text{and} \quad \|\nabla K(x)\| \le \frac{c}{|x|^{n+1}}$$

for some positive constant c depending only n. To this end first observe that by (2.4)

$$|k_{s,t}(x)| \le \frac{cst}{(s+t+|x|)^{n+2}}\,. \tag{2.5}$$

Here as well as in the rest of the paper, letters $c, c', c_1...$ denote positive constants which may depend on n, q, p or B but never on particular functions in consideration. They may also vary from line to line. Let $a \in A$ with $\|a\| \le 1$. Let q' denote the conjugate index of q. Then by the Hölder inequality and (2.5), we deduce

$$\|K(x)(a)(u,s)\|^{q'} \le c^{q'}\int_\Gamma \frac{s^{q'}t^{q'}}{(s+t+|x+u+z|)^{(n+2)q'}}\,\frac{dzdt}{t^{n+1}}\,.$$

Since $|z| < t$, we have

$$\frac{1}{2}\,(s+t+|x+u|) \le s+t+|x+u+z| \le 2(s+t+|x+u|).$$

It then follows that

$$\begin{aligned}\|K(x)(a)(u,s)\|^{q'} &\le c_1^{q'}\int_\Gamma \frac{s^{q'}t^{q'}}{(s+t+|x+u|)^{(n+2)q'}}\,\frac{dzdt}{t^{n+1}}\\ &\le c_2^{q'}\,\frac{s^{q'}}{(s+|x+u|)^{(n+1)q'}}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\|K(x)(a)\|_A^q &= \int_\Gamma \|K(x)(a)(u,s)\|^q\,\frac{duds}{s^{n+1}}\\ &\le c_2^q\int_\Gamma \frac{s^q}{(s+|x+u|)^{(n+1)q}}\,\frac{duds}{s^{n+1}} \le \frac{c_3^q}{|x|^{nq}}\,.\end{aligned}$$

Taking the supremum over all a in the unit ball of A, we deduce that $K(x)$ is a bounded operator on A and

$$\|K(x)\| \le \frac{c_3}{|x|^n}\,.$$

Similarly, we show

$$\|\nabla K(x)\| \le \frac{c_4}{|x|^{n+1}}\,.$$

Therefore, K is a regular vector-valued kernel.

Since Φ is the singular integral operator with kernel K, by [5, Theorem V.3.4] (see also [9, Theorem 4.1]), the lemma is reduced to the boundedness of Φ on $L^p(\mathbb{R}^n; A)$ for some $p \in (1, \infty)$. Clearly, the most convenient choice of p is $p = q$. By (2.3) and the Hölder inequality

$$\|\Phi(h)(x, u, s)\| \le \alpha \cdot \beta,$$

where

$$\begin{aligned}
\alpha^{q'} &= \int_\Gamma \int_{\mathbb{R}^n} |k_{s,t}(x + u + z - y)|\, dy\, \frac{dzdt}{t^{n+1}}\,,\\
\beta^{q} &= \int_\Gamma \int_{\mathbb{R}^n} |k_{s,t}(x + u + z - y)|\, \|h(y, z, t)\|^q\, dy\, \frac{dzdt}{t^{n+1}}\,.
\end{aligned}$$

Using (2.5), we find

$$\begin{aligned}
\alpha^{q'} &\le c \int_\Gamma \int_{\mathbb{R}^n} \frac{st}{(s + t + |x + u + z - y|)^{n+2}}\, dy\, \frac{dzdt}{t^{n+1}}\\
&\le c_1 \int_\Gamma \frac{st}{(s + t)^2}\, \frac{dzdt}{t^{n+1}} \le c_2\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\big\|\Phi(h)\big\|_q^q &= \int_{\mathbb{R}^n} \int_\Gamma \|\Phi(h)(x, u, s)\|^q\, \frac{duds}{s^{n+1}}\, dx\\
&\le c_3 \int_{\mathbb{R}^n} \int_\Gamma \int_\Gamma \int_{\mathbb{R}^n} \frac{st}{(s + t + |x + u + z - y|)^{n+2}}\, dx\, \frac{duds}{s^{n+1}}\, \|h(y, z, t)\|^q\, \frac{dzdt}{t^{n+1}}\, dy\\
&\le c_4 \int_{\mathbb{R}^n} \int_\Gamma \|h(y, z, t)\|^q\, \frac{dzdt}{t^{n+1}}\, dy = c_4 \big\|h\big\|^q_{L^q(\mathbb{R}^n;A)}\,.
\end{aligned}$$

Thus Φ extends to a bounded map on $L^q(\mathbb{R}^n; A)$, so the lemma is proved. □

3 Carleson measures and uniform convexity

The following theorem is the main result of this section. Recall that $\widehat{Q}$ denotes the tent over Q for a cube $Q \subset \mathbb{R}^n$.

Theorem 3.1 *Let B be a Banach space and $2 \le q < \infty$. Then the following statements are equivalent:*

(i) *There exists a positive constant c such that*

$$\Big(\sup_{Q \text{ cube}} \frac{1}{|Q|} \int_{\widehat{Q}} \big(t\, \|\nabla f(x, t)\|\big)^q\, \frac{dxdt}{t}\Big)^{1/q} \le c\, \|f\|_*\,, \quad \forall\, f \in BMO(\mathbb{R}^n; B). \tag{3.1}$$

(ii) *B has an equivalent norm which is q-uniformly convex.*

Inequality (3.1) means that $\big(t\, \|\nabla f(x, t)\|\big)^q\, dxdt/t$ is a Carleson measure on $\mathbb{R}^{n+1}_+$ for every $f \in BMO(\mathbb{R}^n; B)$. In this regard, let us introduce a more function C_q, besides the Lusin function S_q. Given $f : \mathbb{R}^n \to B$ define

$$C_q(f)(x) = \Big(\sup_Q \frac{1}{|Q|} \int_{\widehat{Q}} \big(t\, \|\nabla f(y, t)\|\big)^q\, \frac{dydt}{t}\Big)^{1/q}\,, \tag{3.2}$$

where the supremum runs over all cubes Q containing x. Then (3.1) can be rephrased as

$$\|C_q(f)\|_\infty \le c\,\|f\|_*\,.$$

The proof of Theorem 3.1 and that of Theorem 4.1 below heavily rely on the results on Lusin type and cotype in [9]. We collect them in the following lemma for the convenience of the reader and also for later reference.

Lemma 3.2 *Let B be a Banach space and $2 \le q < \infty$. Then the following statements are equivalent:*

(i) *B is of Lusin cotype q. Namely, for some $p \in (1,\infty)$ (or equivalently, for every $p \in (1,\infty)$) there exists a positive constant c such that*

$$\big\|S_q(f)\big\|_p \le c\,\big\|f\|_p\,, \quad \forall\, f \in L^p(\mathbb{R}^n;B).$$

(ii) *There exists a constant c such that*

$$\big\|S_q(f)\big\|_1 \le c\,\big\|f\|_{H^1_a(\mathbb{R}^n;B)}\,, \quad \forall\, f \in H^1_a(\mathbb{R}^n;B).$$

(iii) *B has an equivalent q-uniformly convex norm.*

(iv) *B^* is of Lusin type q', where q' is the conjugate index of q.*

(v) *B^* has an equivalent q'-uniformly smooth norm.*

Proof of Theorem 3.1. (ii) $\Rightarrow$ (i). Let $f \in BMO(\mathbb{R}^n;B)$ with $\|f\|_* \le 1$. Let $Q \subset \mathbb{R}^n$ be a cube. Set $\widetilde{Q} = 2Q$, the cube of the same center as Q and of double side length. Write

$$f = (f - f_{\widetilde{Q}})1\!\!1_{\widetilde{Q}} + (f - f_{\widetilde{Q}})1\!\!1_{\widetilde{Q}^c} + f_{\widetilde{Q}} \stackrel{\rm def}{=} f_1 + f_2 + f_{\widetilde{Q}}\,.$$

Then

$$\nabla f(x,t) = \nabla f_1(x,t) + \nabla f_2(x,t),$$

so

$$\Big(\frac{1}{|Q|}\int_{\widehat{Q}}\big(t\,\|\nabla f(x,t)\|\big)^q\,\frac{dxdt}{t}\Big)^{1/q} \le \alpha_1 + \alpha_2\,,$$

where

$$\alpha_k = \Big(\frac{1}{|Q|}\int_{\widehat{Q}}\big(t\,\|\nabla f_k(x,t)\|\big)^q\,\frac{dxdt}{t}\Big)^{1/q}\,, \quad k = 1,\,2.$$

For α_1 by the Fubini theorem we have

$$\begin{aligned}
|Q|\,\alpha_1^q &\le c_n^q\int_Q\int_\Gamma\big(t\,\|\nabla f_1(x+z,t)\|\big)^q\,\frac{dzdt}{t^{n+1}}\,dx\\
&= c_n^q\int_Q\big(S_q(f_1)(x)\big)^q dx \le c_n^q\|S_q(f_1)\|_q^q\,,
\end{aligned}$$

where c_n is a constant depending only on n. By (ii) and Lemma 3.2, B is of Lusin cotype q. Thus

$$\|S_q(f_1)\|_q \le c\,\|f_1\|_q\,.$$

However, by the John-Nirenberg theorem

$$\|f_1\|_q \le c'\,|Q|^{1/q}\,\|f\|_* \le c'\,|Q|^{1/q}\,.$$

It then follows that

$$\alpha_1 \leq c_n c\, c' \,.$$

To deal with α_2 we write

$$\nabla f_2(x,t) = \int_{\mathbb{R}^n} \nabla P_t(x-y) f_2(y)dy = \int_{\widetilde{Q}^c} \nabla P_t(x-y) f_2(y)dy.$$

Note that

$$|\nabla P_t(x-y)| \leq \frac{c_n}{(t+|x-y|)^{n+1}}\,.$$

On the other hand, for $(x,t) \in \widehat{Q}$ and $y \in \widetilde{Q}^c$

$$\frac{1}{(t+|x-y|)^{n+1}} \approx \frac{1}{(\ell+|x-y|)^{n+1}}\,,$$

where $\ell = \ell(Q)$ is the side length of Q. Thus

$$\begin{aligned} \|\nabla f_2(x,t)\| &\leq c'_n \int_{\widetilde{Q}^c} \|f_2(y)\| \frac{1}{(\ell+|x-y|)^{n+1}}\, dy \\ &\leq \frac{c''_n}{\ell} \int_{\mathbb{R}^n} \|f_2(y)\|\, P_\ell(x-y)\, dy. \end{aligned}$$

We now use a well-known characterization of BMO functions, in which averages over cubes are replaced by averages against the Poisson kernel. Namely, a function $g : \mathbb{R}^n \to B$ belongs to $BMO(\mathbb{R}^n; B)$ iff

$$\sup_{(x,t)\in\mathbb{R}^n_+} \int_{\mathbb{R}^n} \|g(y) - g(x,t)\|\, P_t(x-y)\, dy < \infty\,.$$

If this is the case, the supremum above is equivalent to $\|g\|_*$ with relevant constants depending only on n. Then we deduce

$$\|\nabla f_2(x,t)\| \leq \frac{c}{\ell}.$$

Therefore,

$$\alpha_2^q \leq \frac{c^q}{\ell^q\,|Q|} \int_{\widehat{Q}} t^q\, \frac{dxdt}{t} \leq c'\,.$$

Combining the preceding inequalities, we find that $\big(t\, \|\nabla f(x,t)\|\big)^q dxdt/t$ is a Carleson measure on $\mathbb{R}^{n+1}_+$ with constant depending only on n, q and B for every $f \in BMO(\mathbb{R}^n; B)$ with $\|f\|_* \leq 1$. This concludes the proof of (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). This proof is harder. Let $A = L^q(\Gamma; B)$ (recall that the cone Γ is equipped with the measure $dzdt/t^{n+1}$). Given a function $f \in L^p(\mathbb{R}^n; B)$ define

$$\mathcal{S}_q(f)(x,z,t) = t\, \frac{\partial}{\partial t} f(x+z,t), \quad x\in\mathbb{R}^n,\ (z,t)\in\Gamma.$$

We regard $\mathcal{S}_q(f)$ as a function on $\mathbb{R}^n$ with values in A. Then

$$\|\mathcal{S}_q(f)(x)\|_A = S^t_q(f)(x)\,,$$

where $S^t_q(f)$ is the Lusin integral function of f but using only the partial derivative in t. Also note that

$$\mathcal{S}_q(f)(x,z,t) = \varphi_t * f(x+z),$$

where φ is defined by (2.1). As in section 2, $\mathcal{S}_q$ can be represented as a singular integral operator with a regular kernel taking values in the space of bounded linear maps from B into A (see [13] for the scalar case and [15] for $\mathbb{T}$). By [9] and[15], (ii) is equivalent to the following inequality

$$\|\mathcal{S}_q(f)\|_* \le c\,\|f\|_\infty\,,\quad \forall\ f\in L^\infty(\mathbb{R}^n;B). \tag{3.3}$$

Note that this inequality is a finite dimensional property. Namely, if (3.3) holds for every finite dimensional subspace E of B in place of B with constant independent of E, then (3.3) holds for the whole B too. Thus we can assume $\dim B<\infty$ in the rest of the proof. To prove (3.3) we will use duality. We first show that (i) implies

$$\|g\|_{H^1_a(\mathbb{R}^n;B^*)}\le c\,\|S^t_{q'}(g)\|_1 \tag{3.4}$$

for all compactly supported continuous functions $g:\mathbb{R}^n\to B^*$. To this end let $f\in BMO(\mathbb{R}^n;B)$ with $\|f\|_*\le 1$. Then by Plancherel's theorem

$$\int_{\mathbb{R}^n}\langle f(x),\ g(-x)\rangle\,dx = 4\int_{\mathbb{R}^{n+1}_+}\langle t\frac{\partial}{\partial t}f(x,t),\ t\frac{\partial}{\partial t}g(-x,t)\rangle\,\frac{dxdt}{t}\,. \tag{3.5}$$

Note that since $\dim B<\infty$, this equality is reduced to the scalar case, in which it is well-known and immediately follows from Plancherel's theorem. Let $C^t_q(f)$ denote the function defined by (3.2) using only the partial derivative in t . Then by (3.1) we find

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}\langle f(x),\ g(-x)\rangle\,dx\Big| &\le 4\int_{\mathbb{R}^{n+1}_+} t\big\|\frac{\partial}{\partial t}f(x,t)\big\|\,t\big\|\frac{\partial}{\partial t}g(-x,t)\big\|\,\frac{dxdt}{t}\\ &\le c'\int_{\mathbb{R}^n} C^t_q(f)(x)\,S^t_{q'}(g)(-x)dx\\ &\le c\,c'\,\|f\|_*\,\|S^t_{q'}(g)\|_1\,,\end{aligned}$$

where we have used Theorem 1 (a) of [4] for the next to the last inequality. Note that the inequality there is proved only for $q=2$; but the arguments can be easily modified to our situation. Taking the supremum over all f in the unit ball of $BMO(\mathbb{R}^n;B)$, we obtain (3.4).

Return back to (3.3). We use again duality, this time that between $BMO(\mathbb{R}^n;A)$ and $H^1_a(\mathbb{R}^n;A^*)$. Fix a function $f\in L^\infty(\mathbb{R};B)$. Recall that $\mathcal{S}_q(f)$ is a function from $\mathbb{R}^n$ to A and the left hand side of (3.3) is $\|\mathcal{S}_q(f)\|_{BMO(\mathbb{R}^n;A)}$. Thus it suffices to prove

$$|\langle\mathcal{S}_q(f),\ h\rangle|\le c\,\|f\|_\infty\,\|h\|_{H^1_a(\mathbb{R}^n;A^*)}\,,\quad\forall\ h\in H^1_a(\mathbb{R}^n;A^*). \tag{3.6}$$

Again by approximation, we need only to consider a nice h. We have

$$\begin{aligned}\langle\mathcal{S}_q(f),\ h\rangle &= \int_{\mathbb{R}^n}\int_\Gamma\langle\varphi_t*f(x+z),\ h(-x,z,t)\rangle\,\frac{dzdt}{t^{n+1}}\,dx\\ &= \int_{\mathbb{R}^n}\int_\Gamma\langle f(y),\ \varphi_t(\,\cdot+z)*h(\,\cdot,z,t)(-y)\rangle\,\frac{dzdt}{t^{n+1}}\,dy\\ &= \int_{\mathbb{R}^n}\langle f(y),\ \Psi(h)(-y)\rangle\,dy,\end{aligned}$$

where

$$\begin{aligned}\Psi(h)(x) &= \int_\Gamma\varphi_t(\,\cdot+z)*h(\,\cdot,z,t)(x)\,\frac{dzdt}{t^{n+1}}\\ &= \int_\Gamma\int_{\mathbb{R}^n}\varphi_t(x+z-y)h(y,z,t)\,dy\,\frac{dzdt}{t^{n+1}}\,.\end{aligned} \tag{3.7}$$

$\Psi(h)$ is a function on $\mathbb{R}^n$ with values in B^*. Therefore, by (3.4)

$$\begin{aligned} |\langle \mathcal{S}_q(f),\ h\rangle| &\le \|f\|_\infty \|\Psi(h)\|_1 \le \|f\|_\infty \|\Psi(h)\|_{H^1_a(\mathbb{R}^n;B^*)} \\ &\le c\,\|f\|_\infty \|S^t_{q'}(\Psi(h))\|_1 = c\,\|f\|_\infty \|\mathcal{S}_{q'}(\Psi(h))\|_{L^1(\mathbb{R}^n;A^*)}\,. \end{aligned}$$

Here we use the same notation $\mathcal{S}$ in the dual setting, which is consistent with the preceding meaning for A^* is the space associated to B^* in the same way as A associated to B:

$$A^* = L^{q'}(\Gamma; B^*).$$

Now it is easy to see that

$$\mathcal{S}_{q'}(\Psi(h)) = \Phi(h),$$

where Φ is the map defined by (2.1) with (B^*, q') instead of (B, q). Thus by Lemma 2.1,

$$\|\mathcal{S}_{q'}(\Psi(h))\|_{L^1(\mathbb{R}^n;A^*)} \le c\,\|h\|_{H^1_a(\mathbb{R}^n;A^*)}\,.$$

Combining the preceding inequalities, we obtain (3.6), and consequently, (3.3) too. This shows the implication (i) $\Rightarrow$ (ii). Thus the proof of the theorem is complete. $\square$

The previous proof of (i) $\Rightarrow$ (ii) shows the following result, which extends [9, Theorem 5.3] (and [15, Theorem 2.5]) to the case $p = 1$.

Corollary 3.3 *Let B be a Banach space and $1 < q \le 2$. Then the following statements are equivalent:*

(i) *B is of Lusin type q.*

(ii) *There exists a constant c such that*

$$\|f\|_{H^1_a(\mathbb{R}^n;B)} \le c\,\|S_q(f)\|_1$$

holds for all compactly supported continuous functions f from $\mathbb{R}^n$ to B.

(iii) *There exists a constant c such that*

$$\|f\|_1 \le c\,\|S_q(f)\|_1$$

holds for all compactly supported continuous functions f from $\mathbb{R}^n$ to B.

4 Carleson measures and uniform smoothness

This section deals with the properties dual to those in Theorem 3.1. The following theorem gives the characterization of Lusin type in terms of Calrseon measures.

Theorem 4.1 *Let B be a Banach space and $1 < q \le 2$. Then the following statements are equivalent:*

(i) *There exists a positive constant c such that*

$$\|f\|_* \le c \left(\sup_{Q \text{ cube}} \frac{1}{|Q|} \int_{\widehat{Q}} \big(t\,\|\nabla f(x,t)\|\big)^q \,\frac{dxdt}{t} \right)^{1/q} \tag{4.1}$$

holds for all compactly supported continuous functions f from $\mathbb{R}^n$ to B.

(ii) *B has an equivalent q-uniformly smooth norm.*

Proof. (ii) $\Rightarrow$ (i) First note that by Lemma 3.2, (ii) is equivalent to

$$\|S_{q'}(g)\|_1 \le c\,\|g\|_{H^1_a(\mathbb{R}^n;B^*)}\,, \quad \forall\, g\in H^1_a(\mathbb{R}^n;B^*). \tag{4.2}$$

Let $f:\mathbb{R}^n\to B$ be a compactly supported continuous function. We are going to prove (4.1). This proof is similar to that of (3.4) but in a converse direction. By approximation, we can assume that f takes values in a finite dimensional subspace of B; then replacing B by this subspace, we can simply assume $\dim B<\infty$. Using the duality between $BMO(\mathbb{R}^n;B)$ and $H^1_a(\mathbb{R}^n;B^*)$, we find a function $g\in H^1_a(\mathbb{R}^n;B^*)$ of unit norm such that

$$\|f\|_*\approx\int_{\mathbb{R}^n}\langle f(x),\ g(-x)\rangle\,dx,$$

where the equivalence constants depend only on n. By approximation, we can further assume that g is sufficiently nice so that all calculations below are legitimate. By Plancherel's theorem

$$\int_{\mathbb{R}^n}\langle f(x),\ g(-x)\rangle\,dx=\int_{\mathbb{R}^{n+1}_+}\langle t\nabla f(x,t),\ t\nabla g(-x,t)\rangle\,\frac{dxdt}{t}\,.$$

By [4] and (4.2), we find

$$\begin{aligned}\int_{\mathbb{R}^n}\langle f(x),\ g(-x)\rangle\,dx &\le \int_{\mathbb{R}^{n+1}_+} t\|\nabla f(x,t)\|\,t\|\nabla g(-x,t)\|\,\frac{dxdt}{t}\\ &\le c'\int_{\mathbb{R}^n} C_q(f)(x)\,S_{q'}(g)(-x)dx\\ &\le c'\|C_q(f)\|_\infty\,\|S_{q'}(g)\|_1\\ &\le c''\|C_q(f)\|_\infty\,\|g\|_{H^1_a(\mathbb{R}^n;B^*)}\le c''\|C_q(f)\|_\infty\,.\end{aligned}$$

Combining the preceding inequalities, we deduce (4.1).

(i) $\Rightarrow$ (ii). Assume (i). It suffices to prove (4.2). We will do this only for the Lusin function involving the partial derivative $\partial/\partial t$, the others being treated similarly. Thus let $S^t_{q'}$ denote this Lusin function. Our task is to show

$$\|S^t_{q'}(g)\|_1 \le c\,\|g\|_{H^1_a(\mathbb{R}^n;B^*)}\,, \quad \forall\, g\in H^1_a(\mathbb{R}^n;B^*). \tag{4.3}$$

We can clearly assume $\dim B<\infty$. Let $A=L^q(\Gamma;B)$ be as in section 2 and keep the notations introduced there. Note that

$$A^*=L^{q'}(\Gamma;B^*).$$

Now fix a nice function $g\in H^1_a(\mathbb{R}^n;B^*)$. Recall that

$$\|S^t_{q'}(g)\|_1=\int_{\mathbb{R}^n}\Big(\int_\Gamma\big(t\|\frac{\partial}{\partial t}g(x+z,t)\|_{B^*}\big)^{q'}\,\frac{dzdt}{t}\Big)^{1/q'}\,dx=\|\widetilde{g}\|_{L^1(\mathbb{R}^n;A^*)}\,,$$

where

$$\widetilde{g}(x,z,t)=t\frac{\partial}{\partial t}g(x+z,t).$$

Thus there exists a function $h\in L^\infty(\mathbb{R}^n;A)$ of norm 1 such that

$$\begin{aligned}\|S^t_{q'}(g)\|_1 &= \int_{\mathbb{R}^n}\int_\Gamma\langle t\frac{\partial}{\partial t}g(x+z,t),\ h(-x,z,t)\rangle\,\frac{dzdt}{t}\,dx\\ &= \int_{\mathbb{R}^n}\langle g(x),\ \Psi(h)(-x)\rangle\,dx,\end{aligned}$$

where Ψ is defined by (3.7). Therefore, by (4.1), we deduce

$$\|S^t_{q'}(g)\|_1 \le c_n \|g\|_{H^1_a(\mathbb{R}^n;B^*)} \|\Psi(h)\|_* \le c_n c \|g\|_{H^1_a(\mathbb{R}^n;B^*)} \|C_q(\Psi(h))\|_\infty .$$

Thus we are reduced to proving

$$\|C_q(\Psi(h))\|_\infty \le c \|h\|_{L^\infty(\mathbb{R}^n;A)}, \quad \forall\, h \in L^\infty(\mathbb{R}^n; A).$$

We will do this only for the partial derivative in the time variable in the gradient. Namely, we have to show

$$\frac{1}{|Q|} \int_{\widehat{Q}} \big(s \,\| \frac{\partial}{\partial s}\, \Psi(h)(x,s)\|\big)^q \,\frac{dxds}{s} \le c^q \|h\|^q_{L^\infty(\mathbb{R}^n;A)} \tag{4.4}$$

for any cube $Q \subset \mathbb{R}^n$. The argument below is similar to the proof of (ii)$\Rightarrow$ (i) in Theorem 3.1. Using φ and $k_{s,t}$ in section 2, we have

$$s\,\frac{\partial}{\partial s}\,\Psi(h)(x,s) = \int_{\mathbb{R}^n}\int_\Gamma k_{s,t}(x+z-y)h(y,z,t)\,\frac{dzdt}{t}\,dy \overset{\text{def}}{=} \widetilde{\Phi}(h)(x,s).$$

Now fix a cube Q and a nice $h \in L^\infty(\mathbb{R}^n; A)$ with $\|h\|_{L^\infty(\mathbb{R}^n;A)} \le 1$. Let $\widetilde{Q} = 2Q$. Decompose h:

$$h = h\mathbb{1}_{\widetilde{Q}} + h\mathbb{1}_{\widetilde{Q}^c} \overset{\text{def}}{=} h_1 + h_2.$$

Then (4.4) is reduced to

$$\beta_k = \left(\frac{1}{|Q|}\int_{\widehat{Q}} \big(\|\widetilde{\Phi}(h_k)(x,s)\|\big)^q\,\frac{dxds}{s}\right)^{1/q} \le c, \quad k = 1,\ 2.$$

β_1 is easy to estimate. Indeed, using the map Φ in (2.2) and Lemma 2.1, we find

$$\begin{aligned} |Q|\,\beta_1^q \ &\le\ c_n^q \int_Q \|\Phi(h_1)(x)\|^q_A\,dx \le c_n^q \|\Phi(h_1)\|^q_{L^q(\mathbb{R}^n;A)} \\ &\le\ c_n^q c^q \|h_1\|^q_{L^q(\mathbb{R}^n;A)} \le c_n^q c^q |Q|; \end{aligned}$$

whence the desired result for β_1. For β_2 a little more effort is needed. By (2.5), we have

$$\|\widetilde{\Phi}(h_2)(x,s)\| \le c \int_{\widetilde{Q}^c}\int_\Gamma \frac{st}{(s+t+|x+z-y|)^{n+2}}\,\|h(y,z,t)\|\,\frac{dzdt}{t^{n+1}}\,dy.$$

By the Hölder inequality and the assumption that $\|h\|_{L^\infty(\Gamma;A)} \le 1$, the internal integral is estimated as follows:

$$\begin{aligned} &\int_\Gamma \frac{st}{(s+t+|x+z-y|)^{n+2}}\,\|h(y,z,t)\|\,\frac{dzdt}{t^{n+1}} \\ &\quad\le \left(\int_\Gamma \frac{(st)^{q'}}{(s+t+|x+z-y|)^{(n+2)q'}}\,\frac{dzdt}{t^{n+1}}\right)^{1/q'} \quad \|h(y)\|_A \\ &\quad\le \left(\int_\Gamma \frac{s^{q'}t^{q'}}{(s+t+|x+z-y|)^{(n+2)q'}}\,\frac{dzdt}{t^{n+1}}\right)^{1/q'} \\ &\quad\approx \frac{s}{(s+|x-y|)^{n+1}}\,. \end{aligned}$$

On the other hand, for $(x,s) \in \widehat{Q}$ and $y \in \widetilde{Q}^c$, we have

$$\frac{s}{(s+|x-y|)^{n+1}} \approx \frac{s}{|x-y|^{n+1}}\,.$$

Therefore,

$$\|\widetilde{\Phi}(h_2)(x,s)\| \le c's \int_{\widetilde{Q}^c} \frac{dy}{|x-y|^{n+1}} \le \frac{c''s}{\ell},$$

where ℓ is the side length of Q. It then follows that $\beta_2 \le c$. Thus (4.4) is proved. This finishes the proof of (4.3), so the implication (i) $\Rightarrow$ (ii) too. $\square$

Proof of Theorem 0.1. Except the difference between $\mathbb{T}$ and $\mathbb{R}$, Theorem 0.1 is a particular case of Theorems 3.1 and 4.1. The proofs of these two latter theorems can be easily adapted to the case of the circle. $\square$

Remark 4.2 The two "if" parts in Theorem 0.1 can be also proved by using the invariance of the expression $\|\nabla f(z)\|^2\, dA(z)$ under Möbius transformations of D. This invariance means that if $w = \varphi(z)$ is a Möbius transformation of D, then

$$\|\nabla f(\varphi(z))\|^2 dA(z) = \|\nabla f(w)\|^2 dA(w).$$

Now assume that B is 2-uniformly convex. Then B is of Lusin cotype 2. Therefore there exists a constant c such that

$$\int_{\mathbb{T}}\int_0^1 (1-r)\|\nabla f(rz)\|^2 dr\, dm(z) \le c\,\|f - f(0)\|_2^2\,, \quad \forall\, f \in L^2(\mathbb{T};B).$$

Then one easily deduces that (with a different c)

$$\int_D (1-|z|^2)\|\nabla f(z)\|^2\, dA(z) \le c\,\|f - f(0)\|_2^2\,.$$

Now let $z_0 \in D$ and let

$$\varphi(z) = \frac{z+z_0}{1+\bar{z}_0 z}\,.$$

Applying the preceding inequality to $f\circ\varphi$, we get

$$\int_D \|\nabla f\circ\varphi(z)\|^2(1-|z|^2)\, dA(z) \le c\,\|f\circ\varphi - f\circ\varphi(0)\|_2^2\,.$$

Then a change of variables and the previous Möbius invariance yield

$$\int_D \|\nabla f(z)\|^2\, \frac{(1-|z|^2)(1-|z_0|^2)}{|1-\bar{z}_0 z|^2}\, dA(z) \le c \int_{\mathbb{T}} \|f(z) - f(z_0)\|^2 P_{z_0}(z) dm(z).$$

Taking the supremum over all $z_0 \in D$ gives the first inequality in Theorem 0.1. The same argument applies to the "if" part in (ii) there. Unfortunately, this simple proof works neither for the case of $q \ne 2$ nor for that of $\mathbb{R}^n$.

We end the paper with some comments on (0.2). If (0.2) holds, then B has an equivalent 2-uniformly convex norm as well as an equivalent 2-uniformly smooth norm. In particular, it is of both cotype 2 and type 2, so isomorphic to a Hilbert space by Kwapień's theorem [7] (see also [12] to which we refer for the notion of type and cotype too). Conversely, if B is isomorphic to a Hilbert space, we get (0.2) as in the scalar case. Let us give a much more elementary argument showing that the validity of (0.2) implies the isomorphism of B to a Hilbert space. The main point is the following remark.

Remark 4.3 Let $1 < q < \infty$ and B be a Banach space. Given a finite sequence $(a_k) \subset B$ consider the function

$$f(z) = \sum_{k\geq 1} a_k z^{2^k} .$$

Then

$$\sup_{z_0\in D} \int_D (1-|z|)^{q-1} \|f'(z)\|^q \, P_{z_0}(z)\, dA(z) \approx \sum_{k\geq 1} \|a_k\|^q \tag{4.5}$$

with universal equivalence constants.

Recall the following well-known (and easily checked) fact

$$\|f\|_* \approx \big\| \sum_{k\geq 1} a_k z^{2^k} \big\|_1 .$$

Combining this with (4.5) we deduce the following result from [2]: If

$$\sup_{z_0\in D} \int_D (1-|z|)^{q-1} \|f'(z)\|^q \, P_{z_0}(z)\, dA(z) \leq c^q \|f\|_*^q$$

holds for any lacunary polynomial f with coefficients in B with some positive constant c, then B is of cotype q; the converse inequality implies that B is of type q.

Let us show (4.5). Since

$$f'(z) = \sum_{k\geq 1} 2^k a_k z^{2^k-1} ,$$

replacing a_k by $2^k a_k$, we see that (4.5) is reduced to

$$\sup_{z_0\in D} \int_D (1-|z|)^{q-1} \|f(z)\|^q \, P_{z_0}(z)\, dA(z) \approx \sum_{k\geq 1} 2^{-qk} \, ||a_k\|^q .$$

The lower estimate is very easy. Indeed, we have (with $z_0 = 0$)

$$\begin{aligned}
\int_D (1-|z|)^{q-1} \|f(z)\|^q \, dA(z) &= \int_0^1 (1-r)^{q-1} \int_{\mathbb{T}} \|f(rz)\|^q \, dm(z)\, rdr \\
&= \sum_{n\geq 1} \int_{1-2^{-n+1}}^{1-2^{-n}} (1-r)^{q-1} \int_{\mathbb{T}} \|f(rz)\|^q \, dm(z)\, rdr \\
&\geq \sum_{n\geq 1} \int_{1-2^{-n+1}}^{1-2^{-n}} (1-r)^{q-1} \, \|a_n\|^q \, r^{q2^n} \, rdr \\
&\approx \sum_{n\geq 1} 2^{-qn} \, \|a_n\|^q .
\end{aligned}$$

For the upper estimate, we first majorize f pointwise. For $n \geq 1$ and $1 - 2^{-n+1} \leq |z| < 1 - 2^{-n}$ we find

$$\|f(z)\| \leq \sum_{k\leq n} \|a_k\| + \sum_{k>n} \|a_k\| \exp(-2^{k-n}) .$$

Let $0 < \alpha < 1$. Then

$$\sum_{k\leq n} \|a_k\| \leq c \, 2^{n\alpha} \big(\sum_{k\leq n} 2^{-k\alpha q} \|a_k\|^q \big)^{1/q} .$$

Similarly, for $\beta > 1$

$$\begin{aligned}\sum_{k>n} \|a_k\| \exp(-2^{k-n}) &\leq \big(\sum_{k>n} 2^{-k\beta q} \|a_k\|^q\big)^{1/q} \big(\sum_{k>n} 2^{k\beta q'} \exp(-q' 2^{k-n})\big)^{1/q'} \\ &\leq c\, 2^{n\beta} \big(\sum_{k>n} 2^{-k\beta q} \|a_k\|^q\big)^{1/q}.\end{aligned}$$

It follows that for any $z_0 \in D$

$$\begin{aligned}\int_D (1-|z|)^{q-1} \|f(z)\|^q\, P_{z_0}(z)\, dA(z) &\leq c \sum_{n\geq 1} 2^{-nq} \Big[2^{nq\alpha} \sum_{k\leq n} 2^{-k\alpha q} \|a_k\|^q + 2^{nq\beta} \sum_{k>n} 2^{-k\beta q} \|a_k\|^q\Big] \\ &\leq c \sum_{k\geq 1} 2^{-qk}\, \|a_k\|^q.\end{aligned}$$

Therefore, (4.5) is proved.

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